On simplicial toric varieties of codimension 2

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Abstract We describe classes of toric varieties of codimension 2 which are either minimally defined by 3 binomial equations over any algebraically closed field, or are set-theoretic complete intersections in exactly one positive characteristic.

Introduction

If K is an algebraically closed field, the minimum number of equations which are needed to define an affine algebraic variety of K^n is called the *arithmetical (ara) rank of* V (or of the defining ideal $I(V)$ of V in the polynomial ring $K[x_1, \ldots, x_n]$). It is well-known that $\operatorname{ara} V \geq \operatorname{codim} V$; if equality holds, V is called a *set-theoretic complete intersection*; more generally, if $\operatorname{ara} V \leq \operatorname{codim} V + 1$, V is called an *almost set-theoretic complete intersection*. Classes of varieties which are (almost) set-theoretic complete intersections where recently considered in several papers by the same author ([1]–[8]). In particular, [8] contains a characterization of all toric varieties which are set-theoretic complete intersections on binomial equations. If $\operatorname{char} K = p > 0$, these are those fulfilling a certain combinatorial property, based on a notion introduced in [15], i.e., the property of being *completely* p*-glued*. This is a sufficient condition for being a set-theoretic complete intersection in characteristic p; it is not known whether it is also necessary, although many examples provide supporting evidence. Classes of toric varieties which are not completely p-glued for any prime p and are not set-theoretic complete intersections in any characteristic where presented in [2]–[4]: the ones treated in [3] and [4] have any codimension greater than 2, those in [2] any codimension greater than or equal to 2. In [6] the authors described a class of toric varieties of codimension 2 which are completely p-glued, and set-theoretic complete intersections in characteristic p, for exactly one prime p; [1] and [5] contain infinitely many such examples in arbitrarily high codimension. In this paper we give sufficient conditions on the parametrization of a toric variety of codimension 2 which assure that it is not a set-theoretic complete intersection in all characteristics different from a given prime $p > 0$. Since, as was shown in [7], every toric variety of codimension 2 is an almost set-theoretic complete intersection, it will follow that the variety has arithmetical rank equal to 3 in these characteristics. This will allow us to find a large class of toric varieties whose arithmetical rank is equal to 3 over any field; it (properly) includes the

toric varieties of codimension 2 considered in [2]. We will also find new examples of toric varieties of codimension 2 in the 5-dimensional affine space which are set-theoretic complete intersections in exactly one positive characteristic.
The set-theoretic complete intersection property in characteristic zero is a much more complex matter. There is an arithmetic criterion on the semigroup which assures that a toric variety is a set-theoretic complete intersection on binomials in characteristic zero: it is obtained from Definition 1 by requiring that $k = 0$. From [8], Theorem 4, we know, however, that the only toric varieties which are set-theoretic complete intersections on binomials in characteristic zero are the complete intersections. Detecting other set-theoretic complete intersections implies finding non-binomial defining equations and is therefore, in general, a difficult task. Eto has recently proven that the toric curve $(t^{17}, t^{19}, t^{25}, t^{27})$ is a set-theoretic intersection on three equations only one of which is binomial [12], whereas it is impossible to find three defining equations two of which are binomial [11].

In this paper $V \subset K^{n+2}$ denotes a toric variety parametrized in the following way:

$$V : \left\{ \begin{array}{rcl} x_1 & = & u_1^d \\ x_2 & = & u_2^d \\ & \vdots & \\ x_n & = & u_n^d \\ y_1 & = & u_1^{a_1} u_2^{a_2} \cdots u_n^{a_n} \\ y_2 & = & u_1^{b_1} u_2^{b_2} \cdots u_n^{b_n} \end{array} \right. ,$$

where d is a positive integer and $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are nonnegative integers such that, for all indices i, either a_i or b_i is non zero. Up to a change of parameters, we may assume that

$$\gcd(d, a_1, \ldots, a_n, b_1, \ldots, b_n) = 1.$$

The form of the first n rows of the parametrization qualifies V as a so-called *simplicial* toric variety.
In [6] we considered the case where d is a prime number p. We first proved that V is completely p-glued, then we characterized the toric varieties V which are not q-glued for any other prime q by giving a necessary and sufficient arithmetic condition on the exponents a_i and b_i.
In this paper d is any positive integer. In Section 1 we assume that d is a power of a prime p and show that then V is completely p-glued (and thus a set-theoretic complete intersection on two binomial equations if $\operatorname{char} K = p$). In Section 2 we give a general condition under which, for every prime divisor p of d, V is not a set-theoretic complete intersection (i.e., $\operatorname{ara} V = 3$) in all characteristics $q \neq p$. We will conclude that, whenever this condition is fulfilled by two different prime divisors p and q of d, then $\operatorname{ara} V = 3$ in all characteristics. The above discussion will settle the problem of the arithmetical rank for many toric varieties in codimension 2, in particular those treated in Section 3. The lower bounds for the arithmetical rank will be provided by cohomological criteria

together with diagram chasing techniques. We will resort to étale cohomology and cohomology with compact support; for the basic notions on this topic we refer to [14] or [13]. The defining equations will be determined by arithmetical tools.
There is a subset T of $\mathbb{N}^n$ attached to V, namely

$$\begin{aligned} T &= \{(d,0,0,\ldots,0),\ (0,d,0,\ldots,0),\ \ldots,\ (0,0,\ldots,d),\\ &\qquad (a_1,a_2,\ldots,a_n),\ (b_1,b_2,\ldots,b_n)\}. \end{aligned}$$

The polynomials in the defining ideal $I(V)$ of V are the linear combinations of binomials

$$B^{\alpha_1^+\alpha_2^+\cdots\alpha_n^+\beta_1^+\beta_2^+}_{\alpha_1^-\alpha_2^-\cdots\alpha_n^-\beta_1^-\beta_2^-} = x_1^{\alpha_1^+}x_2^{\alpha_2^+}\cdots x_n^{\alpha_n^+}y_1^{\beta_1^+}y_2^{\beta_2^+} - x_1^{\alpha_1^-}x_2^{\alpha_2^-}\cdots x_n^{\alpha_n^-}y_1^{\beta_1^-}y_2^{\beta_2^-}$$

with $\alpha_i^+,\alpha_i^-,\beta_i^+,\beta_i^-$ nonnegative integers (not all zero) such that

$$\begin{aligned} \alpha_1^+(d,0,\ldots,0) &+ \alpha_2^+(0,d,0,\ldots,0)+\cdots+\alpha_n^+(0,0,\ldots,0,d)+\\ &\quad \beta_1^+(a_1,a_2,\ldots,a_n)+\beta_2^+(b_1,b_2,\ldots,b_n)=\\ \alpha_1^-(d,0,\ldots,0) &+ \alpha_2^-(0,d,0,\ldots,0)+\cdots+\alpha_n^-(0,0,\ldots,0,d)+\\ &\quad \beta_1^-(a_1,a_2,\ldots,a_n)+\beta_2^-(b_1,b_2,\ldots,b_n). \qquad (*) \end{aligned}$$

There is a one-to-one correspondence between the set of binomials in $I(V)$ and the set of semigroup relations $(*)$ between the elements of T.
Let us recall a combinatorial notion due to Rosales [15], which refers to the subgroup of $\mathbb{Z}^n$ generated by a set T, and is based on the following two definitions, both quoted from [8], pp. 1894–1895.

Definition 1 *Let p be a prime number and let T_1 and T_2 be non-empty subsets of T such that $T = T_1 \cup T_2$ and $T_1 \cap T_2 = \emptyset$. Then T is called a p-gluing of T_1 and T_2 if there are an integer k and a nonzero element $\mathbf{w} \in \mathbb{Z}^n$ such that $\mathbb{Z}T_1 \cap \mathbb{Z}T_2 = \mathbb{Z}\mathbf{w}$ and $p^k\mathbf{w} \in \mathbb{N}T_1 \cap \mathbb{N}T_2$.*

Definition 2 *An affine semigroup $\mathbb{N}T$ is called completely p-glued if T is the p-gluing of T_1 and T_2, where each of the semigroups $\mathbb{N}T_1, \mathbb{N}T_2$ is completely p-glued or a free abelian semigroup.*

We will say that variety V is completely p-glued if so is the corresponding semigroup $\mathbb{N}T$.

1 When V is a set-theoretic complete intersection in characteristic p.

Theorem 1 ([8], Theorem 5, p. 1899) *An affine or projective toric variety of codimension r over a field K of characteristic $p > 0$ is set-theoretically defined by r binomial equations iff it is completely p-glued. In particular, if it is p-glued, it is a set-theoretic complete intersection.*

We can easily describe large classes of completely p-glued toric varieties. We will refer to the variety V introduced above.

Proposition 1 ([8], Example 1) *Suppose that*

$$\mathrm{supp}\,(a_1, a_2, \ldots, a_n) \subset \mathrm{supp}\,(b_1, b_2, \ldots, b_n).$$

Then V is completely p-glued for all primes p (and hence a set-theoretic complete intersection if $\mathrm{char}\, K \neq 0$).

Proposition 2 *Let p be a prime. If $d = p^r$ for some nonnegative integer r, then V is completely p-glued (and hence a set-theoretic complete intersection if $\mathrm{char}\, K = p$).*

Proof . If $r = 0$, then V is, over any field K, a complete intersection on the two binomials

$$F_1 = y_1 - x_1^{a_1} \cdots x_n^{a_n}, \qquad F_2 = y_2 - x_1^{b_1} \cdots x_n^{b_n}.$$

So assume that $r > 0$. Let

$$\begin{aligned} T_1 = \quad & \{(p^r, 0, 0, \ldots, 0),\ (0, p^r, 0, \ldots, 0),\ \ldots,\ (0, 0, \ldots, p^r), \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad (a_1, a_2, \ldots, a_n)\} \end{aligned}$$

and consider

$$T_{11} = \{(p^r, 0, 0, \ldots, 0),\ (0, p^r, 0, \ldots, 0),\ \ldots,\ (0, 0, \ldots, p^r)\},$$

which generates a free abelian semigroup, and

$$T_{12} = \{(a_1, a_2, \ldots, a_n)\}.$$

Then T_1 is the disjoint union of T_{11} and T_{12} and

$$\mathbb{Z}T_{11} \cap \mathbb{Z}T_{12} = \mathbb{Z}\mathbf{v},$$

where

$$\begin{aligned} \mathbf{v} &= p^s(a_1, a_2 \ldots, a_n) \\ &= \frac{a_1}{p^{r-s}}(p^r, 0, 0, \ldots, 0) + \frac{a_2}{p^{r-s}}(0, p^r, 0, \ldots, 0) + \\ &\qquad\qquad \cdots + \frac{a_n}{p^{r-s}}(0, 0, \ldots, p^r) \end{aligned}$$

and p^{r-s} is the maximum power of p which divides a_i for all $i = 1, \ldots, n$. It follows that $\mathbf{v} \in \mathbb{N}T_{11} \cap \mathbb{N}T_{12}$. Therefore, for all primes q, T_1 is the q-gluing of T_{11} and T_{12}. Now, for all nonnegative integers $h \geq s$, we have that $p^{h-s}\mathbf{v} \in \mathbb{N}T_{11} \cap \mathbb{N}T_{12}$, and, more precisely,

$$\begin{aligned} p^{h-s}\mathbf{v} &= p^h(a_1, a_2 \ldots, a_n) \\ &= a_1'(p^r, 0, 0, \ldots, 0) + a_2'(0, p^r, 0, \ldots, 0) + \\ &\qquad\qquad \cdots + a_n'(0, 0, \ldots, p^r), \end{aligned} \tag{1}$$

where we have set $a'_i = \frac{a_i}{p^{r-h}}$ for all indices $i = 1, \ldots, n$. Moreover, let

$$T_2 = \{(b_1, b_2, \ldots, b_n)\}.$$

Then

$$\mathbb{Z}T_1 \cap \mathbb{Z}T_2 = \mathbb{Z}\mathbf{w},$$

where $\mathbf{w} = \lambda(b_1, \ldots, b_n)$, and

$$\lambda = gcd\{k \in \mathbb{N}^* \mid (kb_1, \ldots, kb_n) \in \mathbb{Z}T_1\}.$$

Since $(p^r b_1, \ldots, p^r b_n) \in \mathbb{N}T_1$, it follows that $\lambda = p^t$ for some nonnegative integer $t \leq r$, and $p^{r-t}\mathbf{w} \in \mathbb{N}T_1 \cap \mathbb{N}T_2$. Hence T is the p-gluing of T_1 and T_2 and the variety V given above is completely p-glued. Of course, for all integers $k \geq r$, we have that $p^{k-t}\mathbf{w} \in \mathbb{N}T_1 \cap \mathbb{N}T_2$, i.e.,

$$\begin{aligned} p^{k-t}\mathbf{w} &= p^k(b_1, b_2, \ldots, b_n) \\ &= b'_1(p^r, 0, 0, \ldots, 0) + b'_2(0, p^r, 0, \ldots, 0) + \\ &\qquad \cdots + b'_n(0, 0, \ldots, p^r), \end{aligned} \tag{2}$$

where we have set $b'_i = b_i p^{k-r}$ for all indices $i = 1, \ldots, n$. According to [8], proof of Theorem 2, V is a set-theoretic complete intersection on any pair of binomials:

$$F_1 = y_1^{p^h} - x_1^{a'_1} x_2^{a'_2} \cdots x_n^{a'_n}, \qquad F_1 = y_2^{p^k} - x_1^{b'_1} x_2^{b'_2} \cdots x_n^{b'_n},$$

which are derived from semigroup relations (1) and (2) respectively. In particular, for $h = k = r$ we get the binomials:

$$F_1 = y_1^{p^r} - x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \qquad F_1 = y_2^{p^r} - x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}.$$

Example 1 Consider the following toric variety of codimension 2 in K^5:

$$V : \begin{cases} x_1 = u_1^4 \\ x_2 = u_2^4 \\ x_3 = u_3^4 \\ y_1 = u_1^8 u_3 \\ y_2 = u_2^{12} u_3^3 \end{cases}.$$

Let

$$T_1 = \{(4,0,0),\ (0,4,0),\ (0,0,4),\ (8,0,1)\},$$
$$T_{11} = \{(4,0,0),\ (0,4,0),\ (0,0,4)\}, \qquad T_{12} = \{(8,0,1)\},$$

and

$$T_2 = \{(0, 12, 3)\}.$$

Then $\mathbb{Z}T_{11} \cap \mathbb{Z}T_{12} = \mathbb{Z}(32, 0, 4)$, since, for all integers λ, α, β, equality $\lambda(8, 0, 1) = \alpha(4, 0, 0) + \beta(0, 0, 4)$ implies that $4|\lambda$ and, on the other hand,

$$4(8, 0, 1) = 8(4, 0, 0) + (0, 0, 4). \tag{3}$$

Moreover $\mathbb{Z}T_1 \cap \mathbb{Z}T_2 = \mathbb{Z}(0, 12, 3)$, since

$$(0, 12, 3) = 3(8, 0, 1) - 6(4, 0, 0) + 3(0, 4, 0).$$

On the other hand,

$$4(0, 12, 3) = 12(0, 4, 0) + 3(0, 0, 4), \tag{4}$$

so that $2^2(0, 12, 3) \in \mathbb{N}T_1 \cap \mathbb{N}T_2$, which shows that V is 2-glued. Hence, in characteristic 2, the variety V is a set-theoretic complete intersection on the following two binomials

$$F_1 = y_1^4 - x_1^8 x_3, \qquad F_2 = y_2^4 - x_2^{12} x_3^3,$$

which are derived from semigroup relations (3) and (4) respectively. A complete list of generating binomials for the defining ideal of V is, in every characteristic,

$$y_1^4 - x_1^8 x_3,\ y_2^4 - x_2^{12} x_3^3,\ y_1 y_2 - x_1^2 x_2^3 x_3,$$

$$x_1^4 y_2^2 - x_2^6 x_3 y_1^2,\ x_1^6 y_2 - x_2^3 y_1^3,\ x_1^2 y_2^3 - x_2^9 x_3^2 y_1.$$

In Section 2 we will show that V is not a set-theoretic complete intersection in any characteristic other than 2.

2 When V is not a set-theoretic complete intersection in any characteristic other than p.

In this section we suppose that d is any integer greater than 1.

We assume that the parametrization of V fulfils the following conditions:
(A) there are indices i and j such that

$$a_i = 0, \qquad b_i \neq 0, \qquad \text{and} \qquad a_j \neq 0, \qquad b_j = 0;$$

(B) for all $i = 1, \ldots, n$

(i) $d|a_i \Leftrightarrow d|b_i$;

(ii) $d \not| a_i \Leftrightarrow \gcd(d, a_i) = 1$ and $d \not| b_i \Leftrightarrow \gcd(d, b_i) = 1$;

(C) the matrix of residue classes modulo d

$$\begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \\ \bar{b}_1 & \bar{b}_2 & \cdots & \bar{b}_n \end{pmatrix}$$

has proportional rows, i.e., one is an integer multiple of the other;

(D) there is i such that $\gcd(d, a_i) = 1$.

We will study the set-theoretic complete intersection property for the varieties V fulfilling (A)–(D). We first show that the problem can be reduced to certain hyperplane sections of V, then we resort to étale cohomology. We will first reduce the proof to the case where d does not divide any a_i or b_i for $i \geq 3$ and then we will prove the claim under this additional assumption. The arguments are essentially those used in [6], which are here generalized.
Let an index $i \in \{1, \ldots, n\}$ be fixed. We introduce some abridged notation. For all indices $k = 1, \ldots, n$, we denote by $\mathbf{e}_k$ the kth element of the canonical basis of $\mathbb{Z}^n$, and by $\bar{\mathbf{e}}_k$ the element of $\mathbb{Z}^{n-1}$ obtained by skipping the ith component of $\mathbf{e}_k$. Then $\mathbf{e}_1, \ldots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \ldots, \mathbf{e}_n$ are the elements of the canonical basis of $\mathbb{Z}^{n-1}$. Moreover we set

$$\mathbf{a} = (a_1, a_2, \ldots, a_n), \qquad \text{and} \qquad \mathbf{b} = (b_1, b_2, \ldots, b_n),$$

$$\bar{\mathbf{a}} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n), \text{ and } \bar{\mathbf{b}} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n).$$

We consider the following toric variety in K^{n+1}, whose parametrization is obtained from that of V by omitting the parameter u_i:

$$\bar{V} : \left\{ \begin{array}{rcl} x_1 & = & u_1^d \\ x_2 & = & u_2^d \\ & \vdots & \\ x_{i-1} & = & u_{i-1}^d \\ x_{i+1} & = & u_{i+1}^d \\ & \vdots & \\ x_n & = & u_n^d \\ y_1 & = & u_1^{a_1} u_2^{a_2} \cdots u_{i-1}^{a_{i-1}} u_{i+1}^{a_{i+1}} \cdots u_n^{a_n} \\ y_2 & = & u_1^{b_1} u_2^{b_2} \cdots u_{i-1}^{b_{i-1}} u_{i+1}^{b_{i+1}} \cdots u_n^{b_n} \end{array} \right. .$$

It is associated with the following subset of $\mathbb{N}^{n-1}$:

$$\bar{T} = \{d\bar{\mathbf{e}}_1, \ldots, d\bar{\mathbf{e}}_{i-1}, d\bar{\mathbf{e}}_{i+1}, \ldots, \ d\bar{\mathbf{e}}_n, \bar{\mathbf{a}}, \bar{\mathbf{b}}\}.$$

Lemma 1 *Suppose that d divides both the exponents a_i and b_i in the parametrization of V. Let $F = F(x_1, x_2, \ldots, x_n, y_1, y_2) \in K[x_1, x_2, \ldots, x_n, y_1, y_2]$, and set*

$$\begin{aligned}\bar{F} &= F(x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n, y_1, y_2) \\ &\qquad \in K[x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots x_n, y_1, y_2].\end{aligned}$$

Let $I(V)$ and $I(\bar{V})$ be the defining ideals of V and $\bar{V}$ in $K[x_1, x_2, \ldots, x_n, y_1, y_2]$ and $K[x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y_1, y_2]$ respectively. Then

$$F \in I(V) \Longrightarrow \bar{F} \in I(\bar{V}).$$

Conversely, for all $G \in K[x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y_1, y_2]$ such that $G \in I(\bar{V})$ there is $F \in K[x_1, x_2, \ldots, x_n, y_1, y_2]$ such that $F \in I(V)$ and $\bar{F} = G$.

Proof . It suffices to prove the claim for binomials. Let $B^{\alpha_1^+\alpha_2^+\cdots\alpha_n^+\beta_1^+\beta_2^+}_{\alpha_1^-\alpha_2^-\cdots\alpha_n^-\beta_1^-\beta_2^-}$ be a binomial of $I(V)$. Then the following semigroup relation in T holds:

$$\begin{aligned}\alpha_1^+ d\mathbf{e}_1 + \alpha_2^+ d\mathbf{e}_2 + \cdots + \alpha_n^+ d\mathbf{e}_n + \beta_1^+\mathbf{a} + \beta_2^+\mathbf{b} = \\ \alpha_1^- d\mathbf{e}_1 + \alpha_2^- d\mathbf{e}_2 + \cdots + \alpha_n^- d\mathbf{e}_n + \beta_1^-\mathbf{a} + \beta_2^-\mathbf{b}. \qquad (*)\end{aligned}$$

It follows that $\overline{B^{\alpha_1^+\alpha_2^+\cdots\alpha_n^+\beta_1^+\beta_2^+}_{\alpha_1^-\alpha_2^-\cdots\alpha_n^-\beta_1^-\beta_2^-}} \in I(\bar{V})$, since this binomial corresponds to the following semigroup relation in $\bar{T}$:

$$\begin{aligned}\alpha_1^+ d\bar{\mathbf{e}}_1 + \alpha_2^+ d\bar{\mathbf{e}}_2 + \cdots + \alpha_{i-1}^+ d\bar{\mathbf{e}}_{i-1} + \alpha_{i+1}^+ d\bar{\mathbf{e}}_{i+1} + \cdots + \alpha_n^+ d\bar{\mathbf{e}}_n + \beta_1^+\bar{\mathbf{a}} + \beta_2^+\bar{\mathbf{b}} = \\ \alpha_1^- d\bar{\mathbf{e}}_1 + \alpha_2^- d\bar{\mathbf{e}}_2 + \cdots + \alpha_{i-1}^- d\bar{\mathbf{e}}_{i-1} + \alpha_{i+1}^- d\bar{\mathbf{e}}_{i+1} + \cdots + \alpha_n^- d\bar{\mathbf{e}}_n + \beta_1^-\bar{\mathbf{a}} + \beta_2^-\bar{\mathbf{b}}\ (**)\end{aligned}$$

derived from $(*)$ by skipping the ith component. Conversely, every semigroup relation $(**)$ in $\bar{T}$ gives rise to the following semigroup relation in T:

$$\begin{aligned}\alpha_1^+ d\mathbf{e}_1 + \alpha_2^+ d\mathbf{e}_2 + \cdots + (-\beta_1^+\frac{a_i}{d} - \beta_2^+\frac{b_i}{d})d\mathbf{e}_i + \cdots + \alpha_n^+ d\mathbf{e}_n + \beta_1^+\mathbf{a} + \beta_2^+\mathbf{b} = \\ \alpha_1^- d\mathbf{e}_1 + \alpha_2^- d\mathbf{e}_2 + \cdots + (-\beta_1^-\frac{a_i}{d} - \beta_2^-\frac{b_i}{d})d\mathbf{e}_i + \cdots + \alpha_n^- d\mathbf{e}_n + \beta_1^-\mathbf{a} + \beta_2^-\mathbf{b}.\end{aligned}$$

This proves the second part of the claim.

The following result will be used in the proof of Theorem 2.

Lemma 2 *Suppose that $I(V) = Rad(F_1, \ldots, F_s)$. Then*

$$I(\bar{V}) = Rad(\bar{F}_1, \ldots, \bar{F}_s).$$

Proof . Inclusion $\supset$ follows from Lemma 1, since $I(\bar{V})$ is a reduced ideal. We prove inclusion $\subset$. Let $G \in I(\bar{V})$. By Lemma 1 there is $H \in I(V)$ such that $G = \bar{H}$. Then, for some positive integer m, $H^m \in (F_1, \ldots, F_s)$, i.e.,

$$H^m = \sum_{i=1}^{s} f_i F_i, \qquad \text{for some } f_i \in K[x_1, x_2, \ldots, x_n, y_1, y_2].$$

Since $\bar{f}_i \in K[x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y_1, y_2]$, it follows that

$$G^m = \bar{H}^m = \overline{H^m} = \overline{\sum_{i=1}^{s} f_i F_i} = \sum_{i=1}^{s} \bar{f}_i \bar{F}_i \in (\bar{F}_1, \ldots, \bar{F}_s),$$

which completes the proof.

We will also use the following criterion, cited from [9], Lemma 3′. The symbol H_{et} denotes étale cohomology.

Lemma 3 *Let $W \subset \tilde{W}$ be affine varieties. Let $d = \dim \tilde{W} \setminus W$. If there are s equations $F_1, \ldots, F_s$ such that $W = \tilde{W} \cap V(F_1, \ldots, F_s)$, then*

$$H^{d+i}_{\mathrm{et}}(\tilde{W} \setminus W, \mathbb{Z}/r\mathbb{Z}) = 0 \quad \text{for all } i \geq s$$

and for all $r \in \mathbb{Z}$ which are prime to $\operatorname{char} K$.

The main result of this section is the following:

Theorem 2 *If the variety V introduced above fulfils conditions (A)–(D), and p is any prime divisor of d, then V is not a set-theoretic complete intersection for $\operatorname{char} K \neq p$.*

Proof . Let $\operatorname{char} K \neq p$. By permuting the indices if necessary we can assume that condition (A) takes the form:

$$a_1 = 0, \qquad b_1 \neq 0, \qquad a_2 \neq 0, \qquad b_2 = 0.$$

Condition (B)(i) implies that d divides a_2 and b_1. Hence condition (D) implies that there is $i \geq 3$ such that d is prime to a_i (hence, in view of (B), it is prime to b_i as well). Our aim is to show that V is not set-theoretically defined by two equations. By virtue of Lemma 2 it suffices to show that this is true for the variety $\bar{V}$ whose parametrization is obtained from that of V by omitting all parameters u_i $(3 \leq i \leq n)$ for which $d|a_i$ (equivalently: $d|b_i$). Thus we may assume that in the parametrization of V we have $d \not| a_i$ and $d \not| b_i$ for all indices $i \geq 3$. Then conditions (B) and (D) reduce to the following:

(α) $n \geq 3$ and $\gcd(d, a_i) = \gcd(d, b_i) = 1$ for all $i = 3, 4, \ldots, n$.

Condition (C) takes the form:

(β) in the matrix of residues classes modulo d

$$\begin{pmatrix} \bar{a}_3 & \bar{a}_4 & \cdots & \bar{a}_n \\ \bar{b}_3 & \bar{b}_4 & \cdots & \bar{b}_n \end{pmatrix}$$

either row is an integer multiple of the other.

The variety V has the following parametrization:

$$V : \left\{ \begin{array}{lcl} x_1 & = & u_1^d \\ x_2 & = & u_2^d \\ & \vdots & \\ x_n & = & u_n^d \\ y_1 & = & u_2^{a_2} u_3^{a_3} \cdots u_n^{a_n} \\ y_2 & = & u_1^{b_1} u_3^{b_3} \cdots u_n^{b_n} \end{array} \right. .$$

By Lemma 3 it suffices to show that

$$H^{n+4}_{\mathrm{et}}(K^{n+2} \setminus V, \mathbb{Z}/p\mathbb{Z}) \neq 0.$$

Applying Poincaré Duality (see [13], Cor. 11.2, p. 276) we obtain the equivalent statement:

$$H_c^n(K^{n+2} \setminus V, \mathbb{Z}/p\mathbb{Z}) \neq 0,$$

where H_c denotes cohomology with compact support. For the sake of simplicity, in the sequel we shall omit the coefficient group $\mathbb{Z}/p\mathbb{Z}$. In this and in the following proofs we shall also consider as equal to $\mathbb{Z}/p\mathbb{Z}$ all cohomology groups that are isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Recall that for all nonnegative integers m,

$$H_c^i(K^m) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{for } i = 2m, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Here we have set $K^0 = \{0\}$. In the exact sequence (see [13], Remark 1.30, p. 94)

$$H_c^{n-1}(K^{n+2}) \longrightarrow H_c^{n-1}(V) \longrightarrow H_c^n(K^{n+2} \setminus V) \longrightarrow H_c^n(K^{n+2})$$

we thus have that $H_c^{n-1}(K^{n+2}) = H_c^n(K^{n+2}) = 0$, whence $H_c^n(K^{n+2} \setminus V) \simeq H_c^{n-1}(V)$. Hence we can re-formulate our claim as

$$H_c^{n-1}(V) \neq 0. \tag{6}$$

We prove (6) by induction on n. According to (α) we have $n \geq 3$. Hence the variety to be considered for the initial step of the induction is

$$U : \begin{cases} x_1 &= u_1 \\ x_2 &= u_2 \\ x_3 &= u_3^d \\ y_1 &= u_2^{a_2} u_3^{a_3} \\ y_2 &= u_1^{b_1} u_3^{b_3} \end{cases} .$$

Here we have performed a change of parameters: since d divides a_2 and b_1, we may adjust the parametrization of U by replacing u_1^d, u_2^d, a_2/d and b_1/d by u_1, u_2, a_2 and b_1 respectively. We have to show that

$$H_c^2(U) \neq 0.$$

Now $K[U] = K[u_1, u_2, u_3^d, u_2^{a_2} u_3^{a_3}, u_1^{b_1} u_3^{b_3}] \subset K[u_1, u_2, u_3] = K[K^3]$. This inclusion corresponds to a map

$$\phi : K^3 \to U$$

defined by

$$(u_1, u_2, u_3) \mapsto (u_1, u_2, u_3^d, u_2^{a_2} u_3^{a_3}, u_1^{b_1} u_3^{b_3}),$$

which is a finite (hence a proper) morphism of schemes. Let $X \subset K^3$ be the linear subspace defined by $u_1 = u_2 = 0$. Then X is a one-dimensional affine space. Let $Y = \phi(X)$. We show that ϕ induces by restriction a bijection from $K^3 \setminus X$ to $U \setminus Y$. It suffices to show that for all $(u_1, u_2, u_3^d, u_2^{a_2} u_3^{a_3}, u_1^{b_1} u_3^{b_3})$ such that $u_1 \neq 0$ or $u_2 \neq 0$, u_3 is uniquely determined. This is certainly true if

$u_3 = 0$. Suppose that $u_3 \neq 0$. Since d is, by (α), prime to a_3 and b_3, there are integers v, w, s, t such that

$$vd + wa_3 = 1, \qquad \text{and} \qquad sd + tb_3 = 1.$$

If $u_1 \neq 0$, then

$$u_3 = \frac{(u_3^d)^s (u_1^{b_1} u_3^{b_3})^t}{u_1^{b_1 t}};$$

if $u_2 \neq 0$, then

$$u_3 = \frac{(u_3^d)^v (u_2^{a_2} u_3^{a_3})^w}{u_2^{a_2 w}}.$$

This proves bijectivity. Now let S be the linear subspace of K^3 defined by $u_3 = 0$, and set $T = \phi(S)$. Then ϕ induces by restriction a bijection (in fact, locally an isomorphism) from $K^3 \setminus (X \cup S)$ to $U \setminus (Y \cup T)$. According to [10], Lemma 3.1, bijectivity, together with properness, implies that ϕ induces, for all indices i, an isomorphism between the ith étale cohomology groups of $K^3 \setminus (X \cup S)$ and $U \setminus (Y \cup T)$ with coefficient group $\mathbb{Z}/p\mathbb{Z}$. Since $K^3 \setminus (X \cup S)$ and $U \setminus (Y \cup T)$ are non singular, applying Poincaré Duality we deduce that, for all indices i, ϕ induces an isomorphism

$$H_{\mathrm{c}}^i(K^3 \setminus (X \cup S)) \simeq H_{\mathrm{c}}^i(U \setminus (Y \cup T)).$$

Now, $K^3 \setminus (X \cup S)$ and $U \setminus (Y \cup T)$ are open subsets of $K^3 \setminus X$ and $U \setminus Y = \phi(K^3 \setminus X)$ respectively. Their complements in these spaces can be both identified with the open subset Z of K^2 defined by $u_1 \neq 0$ or $u_2 \neq 0$; the map ϕ induces by restriction the identity map on Z, hence this restriction induces the identity map in cohomology with compact support. Thus ϕ gives rise, for all indices i, to the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
H_{\mathrm{c}}^{i-1}(Z) & \to & H_{\mathrm{c}}^i(U \setminus (Y \cup T)) & \to & H_{\mathrm{c}}^i(U \setminus Y) & \to & H_{\mathrm{c}}^i(Z) & \to & H_{\mathrm{c}}^{i+1}(U \setminus (Y \cup T)) \\
\downarrow \| & & \downarrow \wr & & \downarrow & & \downarrow \| & & \downarrow \wr \\
H_{\mathrm{c}}^{i-1}(Z) & \to & H_{\mathrm{c}}^i(K^3 \setminus (X \cup S)) & \to & H_{\mathrm{c}}^i(K^3 \setminus X) & \to & H_{\mathrm{c}}^i(Z) & \to & H_{\mathrm{c}}^{i+1}(K^3 \setminus (X \cup S))
\end{array}$$

By the Five Lemma it follows that ϕ induces, for all indices i, an isomorphism

$$H_{\mathrm{c}}^i(U \setminus Y) \simeq H_{\mathrm{c}}^i(K^3 \setminus X).$$

In view of (5), from the long exact sequence

$$\begin{array}{ccccccccccc}
H_{\mathrm{c}}^1(X) & \to & H_{\mathrm{c}}^2(K^3 \setminus X) & \to & H_{\mathrm{c}}^2(K^3) & \to & H_{\mathrm{c}}^2(X) & \to & H_{\mathrm{c}}^3(K^3 \setminus X) & \to & H_{\mathrm{c}}^3(K^3) \\
\| & & & & \| & & \| & & & & \| \\
0 & & & & 0 & & \mathbb{Z}/p\mathbb{Z}_p & & & & 0
\end{array}$$

we deduce that $H_{\mathrm{c}}^2(K^3 \setminus X) = 0$ and $H_{\mathrm{c}}^3(K^3 \setminus X) = \mathbb{Z}/p\mathbb{Z}$. Moreover, by [14], Remark 24.2 (f), p. 135, inclusion

$$K[u_3^d] = K[Y] \subset K[X] = K[u_3]$$

induces multiplication by d in cohomology with compact support. Since X and Y are both one-dimensional affine spaces, by (5) this yields the zero map

$$\theta : H_c^2(Y) = \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} = H_c^2(X),$$

and, furthermore,

$$H_c^3(Y) = H_c^3(X) = 0.$$

Therefore, in the morphism of complexes induced by the map ϕ we have the following commutative diagram with exact row:

$$\begin{array}{ccccccc}
 & & \mathbb{Z}/p\mathbb{Z} & & & & \\
 & & \| & f & & & \\
H_c^2(U) & \to & H_c^2(Y) & \to & H_c^3(U \setminus Y) & \to & H_c^3(U) \\
 & & \theta = 0 \downarrow & & \downarrow \wr & & \\
 & & H_c^2(X) & \to & H_c^3(K^3 \setminus X) & & \\
 & & & & \| & & \\
 & & & & \mathbb{Z}/p\mathbb{Z} & &
\end{array}$$

From the commutativity it follows that f must be the zero map, which is not injective. Hence

$$H_c^2(U) \neq 0.$$

This proves the induction basis. Now assume that $n \geq 4$. Since $d \not| a_n$ and $d \not| b_n$, in particular, we have that $a_n \neq 0$ and $b_n \neq 0$. Let W be the intersection of V and the subvariety of K^{n+2} defined by $x_n = 0$. Then W can be identified with K^{n-1}, so that, in view of (5), from the exact sequence

$$\begin{array}{ccccccc}
H_c^{n-2}(W) & \longrightarrow & H_c^{n-1}(V \setminus W) & \longrightarrow & H_c^{n-1}(V) & \longrightarrow & H_c^{n-1}(W) \\
\| & & & & & & \| \\
0 & & & & & & 0
\end{array}$$

we deduce that $H_c^{n-1}(V) \simeq H_c^{n-1}(V \setminus W)$. Hence our claim (6) is equivalent to

$$H_c^{n-1}(V \setminus W) \neq 0. \tag{7}$$

Now, since $n \geq 4$, and, by (α), a_3 is prime to d, we can find a positive integer λ_3 such that d divides $\lambda_3 a_3 + a_4 + \cdots + a_{n-1} + a_n$. On the other hand, by (β), there is an integer μ such that d divides $a_i - \mu b_i$ for all $i \geq 3$. It follows that d divides $\mu(\lambda_3 b_3 + b_4 + \cdots + b_{n-1} + b_n)$, and that μ is prime to d. Hence d divides $\lambda_3 b_3 + b_4 + \cdots + b_{n-1} + b_n$ as well. We conclude that the coordinate ring of $V \setminus W$ is

$$K[V \setminus W] = K[u_n^d, u_n^{-d}] \otimes_K$$

$$K[\tilde{u}_1^d, \tilde{u}_2^d, \tilde{u}_3^d, \ldots, \tilde{u}_{n-1}^d, \tilde{u}_2^{a_2} \tilde{u}_3^{a_3} \cdots \tilde{u}_{n-1}^{a_{n-1}}, \tilde{u}_1^{b_1} \tilde{u}_3^{b_3} \cdots \tilde{u}_{n-1}^{b_{n-1}}],$$

where $\tilde{u}_3 = u_3/u_n^{\lambda_3}$ and $\tilde{u}_i = u_i/u_n$, for all indices $i \neq 3$. Up to renaming the parameters, thus we have

$$\begin{aligned} K[V \setminus W] = \quad & K[u_n^d, u_n^{-d}] \otimes_K \\ & \quad K[u_1^d, \ldots, u_{n-1}^d, u_2^{a_2} u_3^{a_3} \cdots u_{n-1}^{a_{n-1}}, u_1^{b_1} u_3^{b_3} \cdots u_{n-1}^{b_{n-1}}]. \end{aligned}$$

From the Künneth formula for cohomology with compact support ([13], Theorem 8.5, p. 258) we deduce that

$$H_c^{n-1}(V \setminus W) \simeq \bigoplus_{r+s=n-1} H_c^r(K^*) \otimes_K H_c^s(V_1), \tag{8}$$

where we have set $K^* = K \setminus \{0\}$, and $V_1 \subset K^{n+1}$ is the affine toric variety parametrized by

$$V_1 : \left\{ \begin{array}{rcl} x_1 & = & u_1^d \\ x_2 & = & u_2^d \\ x_3 & = & u_3^d \\ & \vdots & \\ x_{n-1} & = & u_{n-1}^d \\ y_1 & = & u_2^{a_2} u_3^{a_3} \cdots u_{n-1}^{a_{n-1}} \\ y_2 & = & u_1^{b_1} u_3^{b_3} \cdots u_{n-1}^{b_{n-1}} \end{array} \right. .$$

Variety V_1 fulfils (α) and (β), therefore the induction hypothesis applies to it. Recall that

$$H_c^i(K^*) = \left\{ \begin{array}{ll} \mathbb{Z}/p\mathbb{Z} & \text{for } i = 1, 2, \\ 0 & \text{otherwise.} \end{array} \right. \tag{9}$$

This, together with (8), implies that

$$H_c^{n-1}(V \setminus W) \simeq H_c^{n-2}(V_1) \oplus H_c^{n-3}(V_1).$$

Now, by the induction hypothesis,

$$H_c^{n-2}(V_1) \neq 0,$$

because this is claim (6) for V_1. This proves (7) and completes the proof of Theorem 2.

In general we have the following result.

Theorem 3 ([7], Theorem 3, p. 889) *Let V be the toric variety defined above. Then V is an almost set-theoretic complete intersection on the three binomials:*

$$\begin{aligned} F_1 &= y_1^{d'} - x_1^{a'_1} x_2^{a'_2} \cdots x_n^{a'_n}, \\ F_2 &= y_2^{d''} - x_1^{b'_1} x_2^{b'_2} \cdots x_n^{b'_n}, \\ F_3 &= M - N y_2^e, \end{aligned}$$

for some suitable monomials M and N and some positive integer e, where we have set $d' = d/\gcd(d, a_1, a_2, \ldots, a_n)$, $d'' = d/\gcd(d, b_1, b_2, \ldots, b_n)$, and $a'_i = a_i/\gcd(d, a_1, a_2, \ldots, a_n)$, $b'_i = b_i/\gcd(d, b_1, b_2, \ldots, b_n)$, for all $i = 1, \ldots, n$. In particular, $2 \leq \mathrm{ara}\, V \leq 3$.

Sections 2 and 3 of [7] contain an explicit construction of M, N and e, which we here briefly sketch. Consider the matrices

$$A_1 = \begin{pmatrix} d & 0 & \cdots & 0 & a_1 \\ 0 & d & \ddots & \vdots & a_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & d & a_n \end{pmatrix}, \qquad A_2 = \begin{pmatrix} d & 0 & \cdots & 0 & a_1 & b_1 \\ 0 & d & \ddots & \vdots & a_2 & b_2 \\ 0 & 0 & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & d & a_n & b_n \end{pmatrix},$$

For $i = 1, 2$, let g_i be the greatest common divisor of all n-minors of A_i. Then set $e = g_1/g_2$, and take any pair of monomials $M, N \in K[x_1, x_2, \ldots, x_n, y_1]$ such that $M - Ny_2^e \in I(V)$; this will be a binomial F_3 fulfilling the claim of Theorem 3.

From Proposition 2, Theorem 2 and Theorem 3 we deduce the next two results.

Corollary 1 *Suppose that the variety V fulfils conditions (A)–(D), and let p be any prime divisor of V. Then*

(i) if $\mathrm{char}\, K \neq p$, then $\mathrm{ara}\, V = 3$;

(ii) if $d = p^r$ for some positive integer r, then $\mathrm{ara}\, V = 2$ for $\mathrm{char}\, K = p$; in particular V is a set-theoretic complete intersection if and only if $\mathrm{char}\, K = p$.

Corollary 2 *If the variety V fulfils conditions (A)–(D) and d has two distinct prime divisors, then $\mathrm{ara}\, V = 3$ in all characteristics, i.e., V is not a set-theoretic complete intersection over any field.*

Remark 1 If we put $r = 1$ in the claim (ii) of Corollary 1 we obtain Theorem 2.1 (c) in [6].

Example 2 Let $V \subset K^{n+2}$ be the simplicial toric variety parametrized as follows:

$$V : \begin{cases} x_1 & = & u_1^{p^r} \\ x_2 & = & u_2^{p^r} \\ & \vdots & \\ x_n & = & u_n^{p^r} \\ y_1 & = & u_1^{p^r k_1} u_3^{a_3} \cdots u_m^{a_m} u_{m+1}^{p^r k_{m+1}} \cdots u_n^{p^r k_n} \\ y_2 & = & u_2^{p^r l_2} u_3^{g a_3} \cdots u_m^{g a_m} u_{m+1}^{p^r l_{m+1}} \cdots u_n^{p^r l_n} \end{cases},$$

where p is a prime, r is a positive integer, $3 \leq m \leq n$, $k_1, k_{m+1}, \ldots, k_n$ and $l_2, l_{m+1}, \ldots, l_n$ are nonnegative integers, and $a_3, \ldots, a_m, g$ are positive integers not divisible by p. Then V fulfils conditions (A)–(D), so that, according to Corollary 1, it is a set-theoretic complete intersection if and only if $\mathrm{char}\, K = p$.

For $m = n = 3$, $p = 2$, $r = 2$, $k_1 = 2$, $a_3 = 1$, $l_2 = 3$, $g = 3$ we obtain the variety V of Example 1; we have thus shown that it is a set-theoretic complete intersection only in characteristic 2.

Example 3 Let $V \subset K^{n+2}$ be the simplicial toric variety parametrized as follows:

$$V : \begin{cases} x_1 &=& u_1^{pqh} \\ x_2 &=& u_2^{pqh} \\ & \vdots & \\ x_n &=& u_n^{pqh} \\ & & \\ y_1 &=& u_1^{pqhk_1} u_3^{a_3} \cdots u_m^{a_m} u_{m+1}^{pqhk_{m+1}} \cdots u_n^{pqhk_n} \\ & & \\ y_2 &=& u_2^{pqhl_2} u_3^{ga_3} \cdots u_m^{ga_m} u_{m+1}^{pqhl_{m+1}} \cdots u_n^{pqhl_n} \end{cases},$$

where p and q are distinct primes, h is a positive integer, $3 \leq m \leq n$, k_1, $k_{m+1}, \ldots, k_n$ and $l_2, l_{m+1}, \ldots, l_n$ are nonnegative integers, and $a_3, \ldots, a_m, g$ are positive integers prime to pqh. Then V fulfils conditions (A)–(D), so that, according to Corollary 2, it is not a set-theoretic complete intersection (i.e., $\operatorname{ara} V = 3$) over any field. This was proven in [2] in the special case where $m = n = 3$ and $a_3 = g = 1$.

In the next section we will give another extension of the class of varieties of codimension 2 considered in [2].

3 Some toric varieties of codimension 2

In this section we will present a class of simplicial toric varieties which are set-theoretic complete intersection in exactly one positive characteristic, or in no characteristic. This class includes the toric varieties of codimension 2 studied in [2]. We will consider the variety $V \subset K^5$ with the following parametrization:

$$V : \begin{cases} x_1 &=& u_1^{d_1} \\ x_2 &=& u_2^{d_2} \\ x_3 &=& u_3^{d_3} \\ y_1 &=& u_1^{a_1} u_3^{a_3} \\ y_2 &=& u_2^{b_2} u_3^{b_3} \end{cases},$$

where $d_1, d_2, d_3, a_1, a_3, b_2, b_3$ are positive integers. Up to a change of parameters, we can assume that

$$\gcd(d_1, a_1) = \gcd(d_2, b_2) = \gcd(d_3, a_3, b_3) = 1. \tag{10}$$

In the main theorem of this section we will give a sufficient criterion on the exponents of the parametrization which assures that V is not a set-theoretic complete intersection (i.e., $\operatorname{ara} V = 3$) in certain characteristics. For the proof we will need the following preliminary results on étale cohomology. They complete Lemma 1 in [3].

Lemma 4 *Let n be a positive integer, and let r be an integer prime to* char K*. Let $d_1, \ldots, d_n$ be positive integers, and consider the morphism of schemes*

$$\gamma_n : K^n \to K^n$$

$$(u_1, \ldots, u_n) \mapsto (u_1^{d_1}, \ldots, u_n^{d_n}),$$

together with its restrictions

$$\delta_n : (K^*)^n \to (K^*)^n,$$

$$\epsilon_n : K \times (K^*)^{n-1} \to K \times (K^*)^{n-1},$$

and the maps

$$\kappa_n : H_{\mathrm{c}}^{n+1}((K^*)^n, \mathbb{Z}/r\mathbb{Z}) \to H_{\mathrm{c}}^{n+1}((K^*)^n, \mathbb{Z}/r\mathbb{Z}),$$

$$\lambda_n : H_{\mathrm{c}}^{n+1}(K \times (K^*)^{n-1}, \mathbb{Z}/r\mathbb{Z}) \to H_{\mathrm{c}}^{n+1}(K \times (K^*)^{n-1}, \mathbb{Z}/r\mathbb{Z}),$$

$$\omega_n : H_{\mathrm{c}}^{n}((K^*)^n, \mathbb{Z}/r\mathbb{Z}) \to H_{\mathrm{c}}^{n}((K^*)^n, \mathbb{Z}/r\mathbb{Z}),$$

induced by δ_n and ϵ_n in cohomology with compact support.

(a) If r is prime to all integers $d_1, \ldots, d_n$, then the maps κ_n and λ_n are isomorphisms.

(b) The map ω_n is an isomorphism.

Proof . In the sequel H_{c} will denote cohomology with compact support with respect to $\mathbb{Z}/r\mathbb{Z}$. We prove (a) by induction on $n \geq 1$. For $n = 1$ we have the morphism

$$\gamma_1 : K \to K$$

$$u_1 \mapsto u_1^{d_1}$$

and its restriction

$$\delta_1 : K^* \to K^*,$$

whereas $\epsilon_1 = \gamma_1$. We know from [14], Remark 24.2 (f), p. 135, that γ_1 induces multiplication by d_1 in cohomology with compact support. Thus, in view of (5) and (9), γ_1 gives rise to the following commutative diagram with exact rows in cohomology with compact support:

$$
\begin{array}{ccccccc}
0 & & \mathbb{Z}/r\mathbb{Z} & & \mathbb{Z}/r\mathbb{Z} & & 0\\
\| & & \| & \simeq & \| & & \|\\
H^1_{\mathrm{c}}(\{0\}) & \to & H^2_{\mathrm{c}}(K^*) & \to & H^2_{\mathrm{c}}(K) & \to & H^2_{\mathrm{c}}(\{0\})\\
 & & \downarrow \kappa_1 & & \lambda_1 \downarrow \cdot d_1 & & \\
H^1_{\mathrm{c}}(\{0\}) & \to & H^2_{\mathrm{c}}(K^*) & \to & H^2_{\mathrm{c}}(K) & \to & H^2_{\mathrm{c}}(\{0\})\\
\| & & \| & \simeq & \| & & \|\\
0 & & \mathbb{Z}/r\mathbb{Z} & & \mathbb{Z}/r\mathbb{Z} & & 0
\end{array}
$$

Since by assumption d_1 and r are coprime, multiplication by d_1 in $\mathbb{Z}/r\mathbb{Z}$, i.e., the map λ_1, is an isomorphism. It follows that κ_1 is an isomorphism as well. Now let $n > 1$ and suppose the claim true for all smaller n. Note that $\{0\} \times K \times (K^*)^{n-2}$ is a closed subset of $K^2 \times (K^*)^{n-2}$ and $(K^2 \times (K^*)^{n-2}) \setminus (\{0\} \times K \times (K^*)^{n-2})$ can be identified with $K \times (K^*)^{n-1}$. After identifying $\{0\} \times K \times (K^*)^{n-2}$ with $K \times (K^*)^{n-2}$ we have the following exact sequence of cohomology with compact support:

$$H_c^n(K^2 \times (K^*)^{n-2}) \to H_c^n(K \times (K^*)^{n-2}) \to H_c^{n+1}(K \times (K^*)^{n-1}) \to H_c^{n+1}(K^2 \times (K^*)^{n-2}). \quad (11)$$

According to the Künneth formula, for all indices i we have

$$H_c^i(K^2 \times (K^*)^{n-2}) \simeq \bigoplus_{s+s_1+\cdots+s_{n-2}=i} H_c^s(K^2) \otimes_K H_c^{s_1}(K^*) \otimes_K \cdots \otimes_K H_c^{s_{n-2}}(K^*).$$

In view of (5) and (9) it follows that $H_c^i(K^2 \times (K^*)^{n-2}) = 0$ for $i < n+2$, in particular

$$H_c^n(K^2 \times (K^*)^{n-2}) = H_c^{n+1}(K^2 \times (K^*)^{n-2}) = 0.$$

Hence γ_n, together with (11), gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \to & H_c^n(K \times (K^*)^{n-2}) & \overset{\simeq}{\to} & H_c^{n+1}(K \times (K^*)^{n-1}) & \to & 0 \\
 & & \downarrow \lambda_{n-1} & & \downarrow \lambda_n & & \\
0 & \to & H_c^n(K \times (K^*)^{n-2}) & \underset{\simeq}{\to} & H_c^{n+1}(K \times (K^*)^{n-1}) & \to & 0
\end{array}$$

Since, by induction, λ_{n-1} is an isomorphism, it follows that λ_n is an isomorphism.

Now, $\{0\} \times (K^*)^{n-1}$ is a closed subset of $K \times (K^*)^{n-1}$ and $(K \times (K^*)^{n-1}) \setminus (\{0\} \times (K^*)^{n-1}) = (K^*)^n$. After identifying $\{0\} \times (K^*)^{n-1}$ with $(K^*)^{n-1}$ we have the following exact sequence of cohomology with compact support:

$$H_c^n(K \times (K^*)^{n-1}) \to H_c^n((K^*)^{n-1}) \to H_c^{n+1}((K^*)^n) \to H_c^{n+1}(K \times (K^*)^{n-1}). \quad (12)$$

According to the Künneth formula, for all indices i we have

$$H_c^i(K \times (K^*)^{n-1}) \simeq \bigoplus_{s+s_1+\cdots+s_{n-1}=i} H_c^s(K) \otimes_K H_c^{s_1}(K^*) \otimes_K \cdots \otimes_K H_c^{s_{n-1}}(K^*). \quad (13)$$

In view of (5) and (9) it follows that

$$H_c^n(K \times (K^*)^{n-1}) = 0. \quad (14)$$

Hence γ_n, together with (12), gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to H^n_{\mathrm{c}}((K^*)^{n-1}) & \to & H^{n+1}_{\mathrm{c}}((K^*)^n) & \to & H^{n+1}_{\mathrm{c}}(K \times (K^*)^{n-1}) \\
\downarrow \kappa_{n-1} & & \downarrow \kappa_n & & \downarrow \lambda_n \\
0 \to H^n_{\mathrm{c}}((K^*)^{n-1}) & \to & H^{n+1}_{\mathrm{c}}((K^*)^n) & \to & H^{n+1}_{\mathrm{c}}(K \times (K^*)^{n-1})
\end{array}$$

Since, by induction, κ_{n-1} is an isomorphism, and so is λ_n by the first part of the proof, by the Four Lemma it follows that κ_n is injective. But, by the Künneth formula and (9), $H^{n+1}_{\mathrm{c}}((K^*)^n)$ is a finite group. Therefore, κ_n is an isomorphism. This completes the proof of (a). Next we prove (b) by induction on $n \geq 1$. In view of (5), the map γ_1 gives rise to the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & & & & & & 0 \\
\| & & & \simeq & & & \| \\
H^0_{\mathrm{c}}(K) & \to & H^0_{\mathrm{c}}(\{0\}) & \to & H^1_{\mathrm{c}}(K^*) & \to & H^1_{\mathrm{c}}(K) \\
 & & \downarrow & & \downarrow \omega_1 & & \\
H^0_{\mathrm{c}}(K) & \to & H^0_{\mathrm{c}}(\{0\}) & \to & H^1_{\mathrm{c}}(K^*) & \to & H^1_{\mathrm{c}}(K) \\
\| & & & \simeq & & & \| \\
0 & & & & & & 0
\end{array}$$

where the leftmost vertical arrow is the identity map, since so is the restriction of γ_1 to $\{0\}$. Hence ω_1 is an isomorphism. Now suppose that $n > 1$ and that ω_{n-1} is an isomorphism. Then from (13), (5) and (9) we have that

$$H^{n-1}_{\mathrm{c}}(K \times (K^*)^{n-1}) = 0.$$

Hence, in view of (14), the map γ_n induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & \simeq & & & & \\
0 & \to & H^{n-1}_{\mathrm{c}}((K^*)^{n-1}) & & \to & & H^n_{\mathrm{c}}((K^*)^n) & \to & 0 \\
 & & \downarrow \omega_{n-1} & & & & \downarrow \omega_n & & \\
0 & \to & H^{n-1}_{\mathrm{c}}((K^*)^{n-1}) & & \to & & H^n_{\mathrm{c}}((K^*)^n) & \to & 0 \\
 & & & & \simeq & & & &
\end{array}$$

from which we conclude that ω_n is an isomorphism. This completes the proof.

Corollary 3 *Let n be a positive integer, and let r be an integer prime to $\operatorname{char} K$. Let X be the subvariety of K^n defined by $x_1 x_2 \cdots x_n = 0$. Consider the following restriction of the morphism γ_n:*

$$\phi_n : X \to X$$

$$(u_1, \ldots, u_n) \mapsto (u_1^{d_1}, \ldots, u_n^{d_n}),$$

and the maps

$$\mu_n : H^n_{\mathrm{c}}(X, \mathbb{Z}/r\mathbb{Z}) \to H^n_{\mathrm{c}}(X, \mathbb{Z}/r\mathbb{Z}),$$
$$\xi_n : H^{n-1}_{\mathrm{c}}(X, \mathbb{Z}/r\mathbb{Z}) \to H^{n-1}_{\mathrm{c}}(X, \mathbb{Z}/r\mathbb{Z})$$

induced by ϕ_n in cohomology with compact support.

> *(a) The map μ_n is an isomorphism if and only if r is prime to all integers $d_1, \ldots, d_n$.*
>
> *(b) The map ξ_n is an isomorphism.*

Proof . In this proof, H_{c} will denote cohomology with compact support with coefficient group $\mathbb{Z}/r\mathbb{Z}$. Since $K^n \setminus X = (K^*)^n$, the morphism γ_n gives rise to the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[d, equal] & & & 0 \arrow[d, equal] \\
H^n_{\mathrm{c}}(K^n) \arrow[r] & H^n_{\mathrm{c}}(X) \arrow[r, "\simeq"] \arrow[d, "\mu_n"] & H^{n+1}_{\mathrm{c}}((K^*)^n) \arrow[r] \arrow[d, "\kappa_n"] & H^{n+1}_{\mathrm{c}}(K^n) \\
H^n_{\mathrm{c}}(K^n) \arrow[r] \arrow[d, equal] & H^n_{\mathrm{c}}(X) \arrow[r, "\simeq"'] & H^{n+1}_{\mathrm{c}}((K^*)^n) \arrow[r] & H^{n+1}_{\mathrm{c}}(K^n) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

It follows that μ_n is an isomorphism if and only if κ_n is. By Lemma 4 the latter condition is true if r is prime to all integers d_i. Otherwise, by Lemma 1 in [3], κ_n is not injective. This proves (a). We also have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[d, equal] & & & 0 \arrow[d, equal] \\
H^{n-1}_{\mathrm{c}}(K^n) \arrow[r] & H^{n-1}_{\mathrm{c}}(X) \arrow[r, "\simeq"] \arrow[d, "\xi_n"] & H^{n}_{\mathrm{c}}((K^*)^n) \arrow[r] \arrow[d, "\omega_n"] & H^{n}_{\mathrm{c}}(K^n) \\
H^{n-1}_{\mathrm{c}}(K^n) \arrow[r] \arrow[d, equal] & H^{n-1}_{\mathrm{c}}(X) \arrow[r, "\simeq"'] & H^{n}_{\mathrm{c}}((K^*)^n) \arrow[r] & H^{n}_{\mathrm{c}}(K^n) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

where ω_n is the isomorphism of Lemma 4. It follows that ξ_n is an isomorphism, too. This completes the proof.

Suppose that the variety V introduced above fulfils the following conditions:

(I) the system of linear congruences

$$\begin{aligned}
a_1x + a_3y &\equiv 0 \quad (\bmod\ d_2)\\
d_1x \quad &\equiv 0 \quad (\bmod\ d_2)\\
d_3y &\equiv 0 \quad (\bmod\ d_2)\\
b_3y &\equiv -b_2 \quad (\bmod\ d_2)
\end{aligned} \tag{15}$$

has a solution;

(II) $d_3' = d_3/\gcd(d_3, a_3)$ is prime to d_1.

We want to give sufficient conditions on the parametrization of this variety V which assure that it is not a set-theoretic complete intersection in all characteristics different from a given prime p. As we have seen in the proof of Theorem 2, this is certainly the case if $H_c^2(V, \mathbb{Z}/p\mathbb{Z}) \neq 0$. This motivates us to discuss the vanishing properties of this cohomology group.

Lemma 5 *Suppose that the variety V introduced above fulfils (I) and (II), and let p be a prime different from $\mathrm{char}\, K$. Then*

$$H_c^2(V, \mathbb{Z}/p\mathbb{Z}) = 0 \qquad \text{if } p \not| d_1d_3',$$

and

$$H_c^2(V, \mathbb{Z}/p\mathbb{Z}) \neq 0 \qquad \text{if } p | d_3' \text{ and } p \not| b_3.$$

Proof . In this proof, H_c will denote cohomology with compact support with coefficient group $\mathbb{Z}/p\mathbb{Z}$. Let W be the intersection of V and the subvariety of K^5 defined by $x_2 = 0$. Then

$$K[W] = K[u_1^{d_1}, u_3^{d_3}, u_1^{a_1}u_3^{a_3}] = K[u_1^{d_1}, u_3^{d_3'}, u_1^{a_1}u_3^{a_3'}],$$

where we have set $a_3' = a_3/\gcd(d_3, a_3)$. Consider the morphism of schemes

$$\phi : K^2 \to W$$

$$(u_1, u_3) \mapsto (u_1^{d_1}, u_3^{d_3'}, u_1^{a_1}u_3^{a_3'}),$$

which is finite, hence proper. Let $X \subset K^2$ be the subvariety defined by $u_1u_3 = 0$. Then $\phi(X) = X$. The morphism ϕ induces by restriction an isomorphism from $K^2\backslash X$ to $W\backslash X$. It suffices to show that for all $(u_1^{d_1}, u_3^{d_3'}, u_1^{a_1}u_3^{a_3'})$ such that $u_1 \neq 0$ and $u_3 \neq 0$, u_1 and u_3 can be expressed as rational functions of $u_1^{d_1}, u_3^{d_3'}, u_1^{a_1}u_3^{a_3'}$. Since d_1 and d_3' are coprime by (II), d_1 is prime to a_1 by (10), and d_3' is prime to a_3' by definition, there are integers v, w, s, t such that

$$vd_1 + wd_3'a_1 = 1, \qquad \text{and} \qquad sd_3' + td_1a_3' = 1.$$

Thus

$$u_1 = \frac{(u_1^{d_1})^v (u_1^{a_1}u_3^{a_3'})^{wd_3'}}{(u_3^{d_3'})^{wa_3'}},$$

and

$$u_3 = \frac{(u_3^{d'_3})^s (u_1^{a_1} u_3^{a'_3})^{td_1}}{(u_1^{d_1})^{ta_1}}.$$

Consequently ϕ induces, for all indices i, an isomorphism of cohomology groups with compact support

$$H^i_{\mathrm{c}}(W \setminus X) \simeq H^i_{\mathrm{c}}(K^2 \setminus X). \tag{16}$$

Note that $K^2 \setminus X = K^* \times K^*$, so that, by the Künneth formula and (9), we have, for all indices i,

$$\begin{aligned} H^i_{\mathrm{c}}(K^2 \setminus X) \quad & \simeq \oplus_{s+t=i} H^s_{\mathrm{c}}(K^*) \otimes_K H^t_{\mathrm{c}}(K^*) \\ & \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{for } i = 2, 4, \\ \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} & \text{for } i = 3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{17}$$

Thus, in view of (5) and (17), we have the following exact sequences:

$$\begin{array}{ccccccc} H^1_{\mathrm{c}}(K^2) & \to & H^1_{\mathrm{c}}(X) & \to & H^2_{\mathrm{c}}(K^2 \setminus X) & \to & H^2_{\mathrm{c}}(K^2) \\ \| & & & & \| & & \| \\ 0 & & & & \mathbb{Z}/p\mathbb{Z} & & 0 \end{array}$$

and

$$\begin{array}{ccccccc} H^0_{\mathrm{c}}(K^2) & \to & H^0_{\mathrm{c}}(X) & \to & H^1_{\mathrm{c}}(K^2 \setminus X) & \to & H^1_{\mathrm{c}}(K^2) \\ \| & & & & \| & & \| \\ 0 & & & & 0 & & 0 \end{array}$$

from which we deduce that $H^1_{\mathrm{c}}(X) = \mathbb{Z}/p\mathbb{Z}$ and $H^0_{\mathrm{c}}(X) = 0$. Finally, from the exact sequence

$$\begin{array}{ccccc} H^0_{\mathrm{c}}(W \setminus X) & \to & H^0_{\mathrm{c}}(W) & \to & H^0_{\mathrm{c}}(X) \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

where we have used (16) and (17), we deduce that

$$H^0_{\mathrm{c}}(W) = 0. \tag{18}$$

Furthermore, in view of (5), (16) and (17), ϕ gives rise to the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[d, equal] & & & & \\ H^1_{\mathrm{c}}(W \setminus X) \arrow[r] & H^1_{\mathrm{c}}(W) \arrow[r] \arrow[d] & H^1_{\mathrm{c}}(X) \arrow[r, "h"] \arrow[d, "\xi_2"] & H^2_{\mathrm{c}}(W \setminus X) \arrow[r] \arrow[d, "\wr"] & H^2_{\mathrm{c}}(W) \arrow[d] \\ & H^1_{\mathrm{c}}(K^2) \arrow[r] \arrow[d, equal] & H^1_{\mathrm{c}}(X) \arrow[r] \arrow[d, equal] & H^2_{\mathrm{c}}(K^2 \setminus X) \arrow[r] \arrow[d, equal] & H^2_{\mathrm{c}}(K^2) \arrow[d, equal] \\ & 0 & \mathbb{Z}/p\mathbb{Z} \arrow[r, phantom, "\simeq"] & \mathbb{Z}/p\mathbb{Z} & 0 \end{tikzcd}$$

and ξ_2 is the map defined in Corollary 3, which is an isomorphism. By the commutativity of the central square it follows that h is an isomorphism, whence

$$H^1_c(W) = 0. \tag{19}$$

We also have the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^1_c(X) \arrow[r] \arrow[d, "\xi_2"', "\wr"] & H^2_c(W\setminus X) \arrow[r] \arrow[d, "\wr"] & H^2_c(W) \arrow[r] \arrow[d, "\bar\phi"] & H^2_c(X) \arrow[r, "k"] \arrow[d, "\mu_2"] & H^3_c(W\setminus X) \arrow[d, "\wr"] & \\
H^1_c(X) \arrow[r] & H^2_c(K^2\setminus X) \arrow[r] \arrow[d, equal] & H^2_c(K^2) \arrow[r] \arrow[d, equal] & H^2_c(X) \arrow[r, "\simeq"'] & H^3_c(K^2\setminus X) \arrow[r] & H^3_c(K^2) \arrow[d, equal] \\
& \mathbb{Z}/p\mathbb{Z} & 0 & & & 0
\end{tikzcd}$$

where ξ_2 and μ_2 are the maps defined in Corollary 3. Hence μ_2 is an isomorphism if and only if $p \not| d_1 d_3'$. In this case, by virtue of the Five Lemma, $\bar\phi$ is an isomorphism, so that $H^2_c(W) = 0$. Otherwise μ_2 is not injective. The commutativity of the right square then implies that k is not injective, so that $H^2_c(W) \neq 0$. We have thus proven that

$$H^2_c(W) = 0 \qquad\qquad \text{if and only if } p \not| d_1 d_3'. \tag{20}$$

Let (s,t) be an integer solution of the equation system (15). Then the coordinate ring of $V \setminus W$ is

$$K[V\setminus W] = K[u_2^{d_2}, u_2^{-d_2}] \otimes_K K[\tilde u_1^{d_1}, \tilde u_3^{d_3}, \tilde u_1^{a_1}\tilde u_3^{a_3}, \tilde u_3^{b_3}],$$

where $\tilde u_1 = u_1^s/u_2$ and $\tilde u_3 = u_3^t/u_2$. Up to renaming the parameters, thus we have

$$K[V\setminus W] = K[u_2^{d_2}, u_2^{-d_2}] \otimes_K K[u_1^{d_1}, u_3^{d_3}, u_1^{a_1}u_3^{a_3}, u_3^{b_3}]. \tag{21}$$

Let $e = \gcd(d_3, b_3)$, and consider the varieties $\tilde W \subset K^4$ and $\bar W \subset K^3$ parametrized in the following ways.

$$\tilde W : \left\{ \begin{array}{lcl} x_1 & = & u_1^{d_1} \\ x_3 & = & u_3^{d_3} \\ y_1 & = & u_1^{a_1}u_3^{a_3} \\ y_2 & = & u_3^{b_3} \end{array} \right. , \qquad \bar W : \left\{ \begin{array}{lcl} x_1 & = & u_1^{d_1} \\ x_3 & = & u_3^{e} \\ y_1 & = & u_1^{a_1}u_3^{a_3} \end{array} \right. .$$

Consider the (finite, proper) morphism of schemes

$$\psi : \bar W \to \tilde W$$

$$(u_1^{d_1}, u_3^{e}, u_1^{a_1}u_3^{a_3}) \mapsto (u_1^{d_1}, u_3^{d_3}, u_1^{a_1}u_3^{a_3}, u_3^{b_3}).$$

Let Y be the intersection of $\bar W$ and the subvariety of K^3 defined by $u_3 = 0$. Then Y is a one-dimensional affine space over K, and the restriction of ψ to Y is the identity map of Y. Moreover, the restriction

$$\psi_{|\bar W\setminus Y} : \bar W \setminus Y \to \tilde W \setminus Y$$

is an isomorphism. Hence it induces isomorphisms in cohomology with compact support. For all indices i the morphism ψ thus gives rise to a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
H_c^{i-1}(Y) & \to & H_c^i(\tilde{W}\setminus Y) & \to & H_c^i(\tilde{W}) & \to & H_c^i(Y) & \to & H_c^{i+1}(\tilde{W}\setminus Y) \\
\downarrow \| & & \downarrow \wr & & \downarrow & & \downarrow \| & & \downarrow \wr \\
H_c^{i-1}(Y) & \to & H_c^i(\bar{W}\setminus Y) & \to & H_c^i(\bar{W}) & \to & H_c^i(Y) & \to & H_c^{i+1}(\bar{W}\setminus Y)
\end{array}$$

From the Five Lemma it follows that the middle vertical arrow is an isomorphism. Hence, for all indices i we have

$$H_c^i(\tilde{W}) \simeq H_c^i(\bar{W}). \tag{22}$$

Note that (II) implies that $\gcd(d_3, d_1)$ divides $\gcd(d_3, a_3)$, whence $\gcd(d_3, b_3, d_1)$ divides $\gcd(d_3, b_3, a_3)$; by (10) it follows that $\gcd(e, d_1) = \gcd(d_3, b_3, d_1) = 1$. Moreover, by (10) we also have that $\gcd(d_1, a_1) = \gcd(e, a_3) = 1$. Therefore, (18), (19) and (20) apply to $\bar{W}$: it suffices to replace a_3' with a_3 and d_3' with e in the argumentation that has been developed above for W. In view of (22) thus follows that

$$\begin{aligned}
H_c^0(\tilde{W}) &= H_c^1(\tilde{W}) = 0, \\
H_c^2(\tilde{W}) &= 0 \quad \text{if and only if} \quad p \not| d_1 e.
\end{aligned} \tag{23}$$

On the other hand, from (21) and the Künneth formula we deduce that, for all indices i,

$$H_c^i(V \setminus W) \simeq \oplus_{s+t=i} H_c^s(K^*) \otimes_K H_c^t(\tilde{W}).$$

Hence, in view of (9) and (23), we conclude that

$$H_c^2(V \setminus W) = 0 \quad \text{and} \quad H_c^3(V \setminus W) = 0 \quad \text{if and only if} \quad p \not| d_1 e. \tag{24}$$

From the exact sequence

$$H_c^2(V \setminus W) \quad \to \quad H_c^2(V) \quad \to \quad H_c^2(W) \quad \to \quad H_c^3(V \setminus W)$$

we deduce that $H_c^2(V) = 0$ if $H_c^2(V \setminus W) = H_c^2(W) = 0$. In view of (20) and (24), we thus have

$$H_c^2(V) = 0 \qquad \text{if } p \not| d_1 d_3'.$$

We also deduce that $H_c^2(V) \neq 0$ if $H_c^2(W) \neq 0$ and $H_c^3(V \setminus W) = 0$, which, in view of (20) and (24), occurs if $p \not| d_1 e$ and $p | d_3'$. Since d_1 and d_3' are coprime by assumption (II), the latter condition implies that $p \not| d_1$. Moreover, since $p | d_3$, $p \not| e$ is equivalent to $p \not| b_3$. Hence we have that

$$H_c^2(V) \neq 0 \qquad \text{if } p | d_3' \text{ and } p \not| b_3.$$

This completes the proof.

Theorem 4 *If the variety V introduced above fulfils conditions (I) and (II) and p is a prime divisor of d_3' and not of b_3, then $\operatorname{ara} V = 3$ for $\operatorname{char} K \neq p$. In particular, if d_3' has two distinct prime divisors not dividing b_3, then $\operatorname{ara} V = 3$ over every field.*

Proof . As in the proof of Theorem 2, it suffices to prove that, under the given assumption, if $\operatorname{char} K \neq p$, then

$$H^2_{\mathrm{c}}(V, \mathbb{Z}/p\mathbb{Z}) \neq 0.$$

But this follows from Lemma 5. This completes the proof.

Example 4 Theorem 4 allows us to find new examples of toric varieties which are set-theoretic complete intersections in exactly one positive characteristic, in addition to those presented in [1], [5] and [6]. Let p and q be distinct primes and consider the variety

$$V : \begin{cases} x_1 &= u_1^{d_1} \\ x_2 &= u_2^{q} \\ x_3 &= u_3^{pq} \\ y_1 &= u_1^{a_1}u_3^{cq} \\ y_2 &= u_2^{b_2}u_3^{b_3} \end{cases},$$

where $\gcd(d_1, a_1) = 1$, d_1 and c are not divisible by p, b_2 is not divisible by q, and b_3 is not divisible by p and q. If

$$T_1 = \{(d_1, 0, 0),\ (0, q, 0),\ (0, 0, pq),\ (a_1, 0, cq)\},$$

$$T_{11} = \{(d_1, 0, 0),\ (0, q, 0),\ (0, 0, pq)\}, \qquad T_{12} = \{(a_1, 0, cq)\},$$

and

$$T_2 = \{(0, b_2, b_3)\},$$

then $\mathbb{Z}T_{11} \cap \mathbb{Z}T_{12} = \mathbb{Z}d_1p(a_1, 0, cq)$, since $\gcd(d_1, a_1) = 1$ and p does not divide d_1 nor cq. In fact

$$d_1p(a_1, 0, cq) = a_1p(d_1, 0, 0) + d_1c(0, 0, pq) \in \mathbb{N}T_{11} \cap \mathbb{N}T_{12}. \qquad (25)$$

Now, for every integer α, $\mathbb{Z}\alpha(0, b_2, b_3) \in \mathbb{Z}T_1 \cap \mathbb{Z}T_2$ holds if and only if there are integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\alpha(0, b_2, b_3) = \alpha_1(d_1, 0, 0) + \alpha_2(0, q, 0) + \alpha_3(0, 0, pq) + \alpha_4(a_1, 0, cq),$$

i.e.,

$$\begin{aligned} 0 &= \alpha_1 d_1 + \alpha_4 a_1, & (26)\\ \alpha b_2 &= \alpha_2 q, & (27)\\ \alpha b_3 &= \alpha_3 pq + \alpha_4 cq. & (28) \end{aligned}$$

From (27) we deduce that q divides α, because q does not divide b_2. On the other hand, since p does not divide d_1c, there are integers λ, μ such that $b_3 = \lambda p + \mu d_1 c$, whence

$$qb_3 = \lambda pq + \mu d_1 cq.$$

Hence (26), (27) and (28) are fulfilled for $\alpha = q$, $\alpha_1 = -\mu a_1$, $\alpha_2 = b_2$, $\alpha_3 = \lambda$, $\alpha_4 = \mu d_1$. Thus

$$\mathbb{Z}T_1 \cap \mathbb{Z}T_2 = \mathbb{Z}q(0, b_2, b_3).$$

But

$$pq(0, b_2, b_3) = b_2 p(0, q, 0) + b_3(0, 0, pq) \in \mathbb{N}T_1 \cap \mathbb{N}T_2, \tag{29}$$

which, together with (25), shows that V is completely p-glued. Hence, for $\operatorname{char} K = p$, V is a set-theoretic complete intersection on the two binomials

$$F_1 = y_1^{d_1 p} - x_1^{a_1 p} x_3^{d_1 c}, \qquad F_2 = y_2^{pq} - x_2^{b_2 p} x_3^{b_3},$$

which are derived from semigroup relations (25) and (29) respectively. Since $d_3' = p$ is prime to d_1, we also have that (II) is fulfilled. Since b_3 and $d_2 = q$ are coprime, there is an integer y such that $b_3 y \equiv -b_2 \pmod{q}$. Then (q, y) is a solution of (15), so that (I) is fulfilled, too. Hence, by Theorem 4, V is not a set-theoretic complete intersection, i.e., $\operatorname{ara} V = 3$, if $\operatorname{char} K \neq p$.

Example 5 Theorem 4 also allows us to find new examples of toric varieties which are not set-theoretic complete intersections in any characteristic, in addition to those presented in [2]. Let

$$V : \begin{cases} x_1 = u_1^{d_1} \\ x_2 = u_2 \\ x_3 = u_3^{d_3} \\ y_1 = u_1^{a_1} u_3^{a_3} \\ y_2 = u_2^{b_2} u_3^{b_3} \end{cases},$$

where $\gcd(d_1, a_1) = \gcd(d_1, d_3) = 1$, d_3 is divisible by two distinct primes p and q, and p and q do not divide a_3 nor b_3. Then (I) and (II) are trivially fulfilled; since p and q divide d_3', by Theorem 4 it follows that V is not a set-theoretic complete intersection over any field, i.e., $\operatorname{ara} V = 3$ over any field. For $d_1 = a_3 = b_3 = 1$ we obtain the varieties presented in [2].

Example 6 Let p be a prime, r a positive integer and consider the variety

$$V : \begin{cases} x_1 &= u_1^{p^r} \\ x_2 &= u_2 \\ x_3 &= u_3 \\ y_1 &= u_1^{a_1} u_3^{a_3} \\ y_2 &= u_2^{b_2} u_3^{b_3} \end{cases},$$

where a_1, a_3, b_2, b_3 are arbitrary positive integers. Then (I) and (II) are fulfilled, and, by Lemma 5, for all primes $q \neq p$, we have that $H_c^2(V, \mathbb{Z}/q\mathbb{Z}) = 0$ if $\operatorname{char} K \neq q$. This means that our cohomological criterion for V being not a set-theoretic complete intersection in all characteristics different from q does not apply. In fact, we obtain an equivalent parametrization for V if we replace u_2 and u_3 with $u_2^{p^r}$ and $u_3^{p^r}$ respectively; then the parametrization takes the form considered in Proposition 2, which allows us to conclude that V is a set-theoretic complete intersection if $\operatorname{char} K = p$. If $r = 0$, then V is a complete intersection over every field K. It can be easily shown that we have, at the same time, (10), condition (I), and, with respect to the notation of Lemma 5, $p \not| d_1 d_3'$ for every prime p, if and only if, up to a change of parameters, $d_1 = 1$, d_2 divides d_3 and d_3 divides a_3. In this case V is a complete intersection on

$$F_1 = y_1 - x_1^{a_1} x_3^{a_3/d_3}, \qquad F_2 = y_2^{d_3} - x_2^{b_2 d_3/d_2} x_3^{b_3}.$$

Final Remark All the examples of toric varieties presented here and in the papers quoted in the references are either

- p-glued for every prime p, or
- p-glued for exactly one prime p, or
- not p-glued for any prime p.

In all the cases where we could determine the arithmetical rank in all prime characteristics, it turned out that the variety is, respectively,

- a set-theoretic complete intersection on binomials in every prime characteristic p, or
- a set-theoretic complete intersection (on binomials) in exactly one prime characteristic p, or
- not a set-theoretic complete intersection in any prime characteristic p.

This leaves the following questions open.

(1) Is there any toric variety which is p-glued for two distinct primes p, but not for all primes p?

(2) Is there any toric variety which is a set-theoretic complete intersection, but not on binomials, in some prime characteristic p?

(3) Is there any toric variety which is a set-theoretic complete intersection in two different prime characteristics, but not in all prime characteristics p?

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